

**FROM QUANTUM MONODROMY TO DUALITY**César Gómez<sup>1,2</sup> and Esperanza López<sup>2</sup>*TH-Division, CERN,  
CH-1211 Geneva 23, Switzerland***Abstract**

For  $N = 2$  SUSY theories with non-vanishing  $\beta$ -function and one-dimensional quantum moduli, we study the representation on the special coordinates of the group of motions on the quantum moduli defined by  $\Gamma_W = Sl(2; Z)/\Gamma_M$ , with  $\Gamma_M$  the quantum monodromy group.  $\Gamma_W$  contains both the global symmetries and the strong-weak coupling duality. The action of  $\Gamma_W$  on the special coordinates is not part of the symplectic group  $Sl(2; Z)$ . After coupling to gravity, namely in the context of non-rigid special geometry, we can define the action of  $\Gamma_W$  as part of  $Sp(4; Z)$ . To do this requires singular gauge transformations on the "scalar" component of the graviphoton field. In terms of these singular gauge transformations the topological obstruction to strong-weak duality can be interpreted as a  $\sigma$ -model anomaly, indicating the possible dynamical role of the dilaton field in  $S$ -duality.

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1. *Introduction.* Given an  $N = 2$  supersymmetric gauge theory, the geometry of the moduli parametrizing the different vacuum expectation values allowed by the flat potential, as a consequence of the non-renormalization theorems [1], is determined by all the quantum corrections. The exact quantum moduli for  $N=2$ ,  $SU(2)$  pure Yang-Mills, was first obtained by Seiberg and Witten in reference [2], extended to  $N = 2$  SQCD- $SU(2)$  with  $N_f \leq 4$  in [3] and to  $N = 2$   $SU(N_c)$  pure Yang-Mills in references [4, 5]. In all these solutions a beautiful geometrical picture emerges. Namely associated with the four-dimensional theory there exists a hyperelliptic curve  $\Sigma_U$ , of genus  $r$  equal to the rank of the gauge group, parametrized by the quantum moduli, whose points we denote as  $U = (u_1, \dots, u_r)$ .

For  $N = 2$  supersymmetry the geometry of the quantum moduli is forced to be rigid special Kähler [6], which implies, for a gauge group of rank  $r$ , the existence of  $2r$  holomorphic sections  $(a_i(U), a_{Di}(U))$   $i = 1, \dots, r$  of the  $Sl(2r; \mathbb{Z})$  bundle defined by the first homology group  $H^1(\Sigma_U; \mathbb{C})$  of the curve  $\Sigma_U$ . The physical spectrum is given by the mass formula

$$\begin{aligned} M &= \sqrt{2}|Z| \\ Z &= \sum_{i=1}^r (n_i^e a_i(U) + n_i^m a_{Di}(U)) \end{aligned} \quad (1)$$

where  $n_i^e$  and  $n_i^m$  are the electric and magnetic charges respectively, and the sections  $(a_i(U), a_{Di}(U))$  can be represented as the periods of some meromorphic 1-form  $\lambda$  over a basis of 1-cycles  $\gamma_i, \tilde{\gamma}_i$

$$a_i = \oint_{\gamma_i} \lambda, \quad a_{Di} = \oint_{\tilde{\gamma}_i} \lambda \quad (2)$$

The mass formula (1,2) already implies that when the curve degenerates some particle in the spectrum can become massless.

Reducing ourselves to the elliptic case, the metric on the quantum moduli, given by the rigid special Kähler relation

$$\tau(u) = \frac{da_D/du}{da/du} \quad (3)$$

turns out to be the elliptic modulus of the curve  $\Sigma_u$ . The function  $\tau(u)$  defined by (3) is the F-term of the low energy lagrangian and can therefore be used to define the wilsonian effective coupling and  $\theta$ -parameter as follows

$$\tau(u) = i \frac{4\pi}{g_{eff}^2(u)} + \frac{\theta_{eff}(u)}{2\pi} \quad (4)$$

Being  $\tau(u)$  the modulus of an elliptic curve, the positivity of the coupling constant is automatically assured. Moreover the Montonen-Olive [7] duality transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2; \mathbb{Z}) \quad (5)$$

coincide with the modular group of the elliptic curve. Defining the curve  $\Sigma_u$  by the vanishing locus of a cubic polynomial in  $P^2$

$$W(x, y, z; u) = 0 \quad (6)$$

the modular group  $Sl(2; Z)$ , of the elliptic curve defined by (6), appears naturally decomposed into two pieces<sup>3</sup> : *i*) the group  $\Gamma_M$ <sup>4</sup> of monodromy transformations around the singularities in the  $u$ -plane, and *ii*) the group  $\Gamma_W$  of the coordinate transformations satisfying

$$W(x', y', z'; u') = f(u)W(x, y, z; u) \quad (7)$$

i.e. transformations on the "target" coordinates which can be, up to a global factor, compensated by a change in the quantum moduli coordinate  $u$ . The explicit relation between  $\Gamma_M$ ,  $\Gamma_W$  and  $Sl(2; Z)$ , which is known in the context of Landau-Ginzburg theories [10], is

$$\Gamma_W = \frac{Sl(2; Z)}{\Gamma_M} \quad (8)$$

For generic  $N = 2$  theories the Montonen-Olive duality (5) is lost, mainly because the  $\beta$ -function is non-vanishing and that electrically and magnetically charged particles transform in different representations under supersymmetry. Nevertheless it was shown in [2, 3] that the monodromy subgroup  $\Gamma_M$  of the  $Sl(2; Z)$  transformations (5), is actually an exact symmetry of the quantum theory. This is in general a non-perturbative symmetry if the monodromy subgroup, as is the case for the examples in [2, 3], contains elements with entry  $c \neq 0$ . The fact that the  $N = 2$  theory is only dual with respect to the monodromy subgroup means, in particular, that the four-dimensional physics depends not only on the moduli of the curve but also on its geometry. This is quite different to what we are used to find in string theory, where the string only feels the moduli (complex or Kähler) of the target space<sup>5</sup>, the difference being the non-vanishing  $\beta$ -function for the  $N = 2$  theory.

*2. The meaning of  $\Gamma_W$ .* On the quantum moduli is also defined the action of the global  $U(1)_{\mathcal{R}}$ -symmetries which are broken to some discrete group by instanton effects, so  $Z_2$  for pure  $SU(2)$  Yang-Mills and  $Z_3$ ,  $Z_2$  for massless SQCD with  $N_f = 1, 2$  respectively. These global symmetries are automatically part of the group  $\Gamma_W$ .

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<sup>3</sup>The analysis we are using here is the algebraic approach to the moduli problem. This approach is familiar in the study of mirror symmetry, see for instance [8]. In that case  $\tau(u)$  will have the meaning of the mirror map.

<sup>4</sup>The monodromy group  $\Gamma_M$  is the monodromy group of the Picard-Fuchs equation for the cycles of the curve  $\Sigma_u$  [9].

<sup>5</sup>Notice that if we consider, following reference [4], the formal type II string whose target space is defined by multiplying the algebraic curves defining the quantum moduli, then for this string,  $Sl(2; Z)$  will be its target space duality and it will contain both  $\Gamma_M$  and  $\Gamma_W$ .

Each element  $\gamma \in \Gamma_W$  is acting on the quantum moduli by  $\gamma(u) = u'$ , where  $u$  and  $u'$  are determined by equation (7). It is clear from the definition of  $\Gamma_W$  that  $\gamma(p_i) = p_j$  for  $p_i, p_j$  singular points in the  $u$ -plane. The role of  $\Gamma_W$  in the characterization of the monodromies around the different singularities is as follows. For each singular point  $p_i$  we can choose *local* special coordinates in such a way that in the neighbourhood of the singularity,  $a_D(u)$  is determined by the one-loop contribution of the particles becoming massless at that singular point. The monodromy of the so defined  $a_D(u)$  function will be  $T^{k_i}$ , for some  $k_i$  depending on the quantum numbers of the particles that become massless at that singular point. Now we can look for the element  $\gamma_i \in \Gamma_W$  such that  $\gamma(\infty) = p_i$ . Then the monodromy  $M_i$  around  $p_i$  will be given by

$$M_i = \Gamma_{\gamma_i} T^{k_i} \Gamma_{\gamma_i}^{-1} \quad (9)$$

where

$$\tau(\gamma_i(u)) = \frac{a\tau(u) + b}{c\tau(u) + d}, \quad \Gamma_{\gamma_i} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (10)$$

Notice that (10) reflects the fact that two points  $u, u'$  related by any element in  $\Gamma_W$  correspond to the same complex structure of the curve  $\Sigma_u$ . Equation (9) clarifies the physical meaning of  $\Gamma_W$ , at least the part of  $\Gamma_W$  which maps the singularity at  $\infty$  into the rest of the singularities. In fact this part of  $\Gamma_W$  relates the local weak coupling description around the singular points  $p_i$  with the original coordinates used in the description of the asymptotically free weak coupling regime at  $\infty$ . Therefore they play the role of defining the dual weak coupling description of a naturally strong coupling regime. This form of duality is crucial when we want to argue that the monodromy subgroup is actually an exact symmetry. In fact in the appropriate dual variables, the monodromy is always a  $T$ -transformation, which only changes  $\Theta_{eff}/2\pi$  by an integer number. We observe, in consequence, that the curve  $\Sigma_u$  contains in a natural and unified way both the information about the non-perturbative symmetries of the physical system  $\Gamma_M$ , and about its dual strong-weak coupling descriptions, enclosed in  $\Gamma_W$ .

*3. The action of  $\Gamma_W$  on the special coordinates.* Let us now consider more closely why the Montonen-Olive duality (5) is actually broken to  $\Gamma_M$ . This will be another way to see the dependence of the four-dimensional physics on the geometry of the curve  $\Sigma_u$  and not only on its moduli. In order to do that, we will consider an element  $\gamma_i \in \Gamma_W$  such that

$$\tau(\gamma_i(u)) = -\frac{1}{\tau(u)} \quad (11)$$

and we will compare  $a(\gamma_i(u))$  with  $a_D(u)$ . More precisely we will lift to the bundle the action of  $\Gamma_W$  on the  $u$ -plane. Notice that if  $a(\gamma_i(u)) = a_D(u)$  we would get a strictly strong-weak duality, namely that the physics in  $\gamma_i(u)$  is dual to the physics in  $u$ , where  $\gamma_i(u)$  and  $u$  correspond, by (4) and (11), to strong and weak coupling regimes respectively.

We will consider first the case of pure  $SU(2)$  Yang-Mills. The exact solution for the quantum moduli is given by the elliptic curve [2]

$$y^2 = (x + \Lambda^2)(x - \Lambda^2)(x - u) \quad (12)$$

which becomes singular at  $u = \pm\Lambda^2, \infty$ , and where  $\Lambda$  is the dynamically generated scale. The monodromy around the singularities generates the group  $\Gamma_2$  of unimodular matrices congruent to 1 modulo 2. The group  $\Gamma_W$  in this case is the dihedral group of six elements [11] ( $[Sl(2; Z) : \Gamma_2] = 6$ ), which on the  $u$ -plane interchanges the three singularities. Let us consider  $\Lambda^2 = 1$  in order to simplify notation, then  $\Gamma_W$  is given by

$$\begin{aligned} u &\rightarrow -u \\ u &\rightarrow \frac{u+3}{u-1} \end{aligned} \quad (13)$$

The transformation (13.1) is the part of  $\Gamma_W$  corresponding to the global  $U(1)_{\mathcal{R}}$ -symmetry spontaneously broken to  $Z_2$ , while the transformation (13.2) maps the singularity at  $\infty$  into the point  $u = 1$ . The fact that any two points of the quantum moduli related by an element in  $\Gamma_W$  correspond to the same complex structure of the curve  $\Sigma_u$ , together with the definitions (2), (3) of the sections  $(a, a_D)$  as periods of some 1-form  $\lambda$  and of the elliptic modulus  $\tau$ , imply that

$$\frac{d\lambda}{du} \sim \lambda_1 \quad (14)$$

with the proportionality factor determined by the asymptotic behaviour and where  $\lambda_1 = dx/y$  is the unique everywhere non-zero holomorphic 1-form, in terms of which the parameter  $\tau$  is represented by

$$\tau = \frac{b_2}{b_1} \quad , \quad b_i = \oint_{\gamma_i} \lambda_1 \quad , \quad i = 1, 2 \quad (15)$$

The solution [2] for  $\lambda$  is

$$\lambda = \frac{\sqrt{2}}{2\pi} (\lambda_2 - u\lambda_1) \quad (16)$$

with  $\lambda_2 = xdx/y$ . Defining  $u' = \gamma(u) = \frac{u+3}{u-1}$  and  $x'(x, u)$  as in (7), we observe that

$$\tau(\gamma(u)) = -\frac{1}{\tau(u)} \quad (17)$$

where we have used

$$\lambda(x', \gamma(u)) \equiv \lambda^\gamma(x, u) = \frac{\sqrt{2}}{2\pi} \left( \frac{2}{1-u} \right)^{1/2} \frac{dx\sqrt{x+1}}{\sqrt{(x-1)(x-u)}} \quad (18)$$

The lift of the action of  $\gamma$  on the holomorphic sections is then given by

$$\begin{aligned} a(\gamma(u)) &= \oint_{\gamma_{1'}} \lambda^\gamma(u) = -\left(\frac{2}{1-u}\right)^{1/2} \left(a_D(u) + \frac{u+1}{\sqrt{2\pi}} b_2(u)\right) \equiv a_D^\gamma(u) \\ a_D(\gamma(u)) &= \oint_{\gamma_{2'}} \lambda^\gamma(u) = -\left(\frac{2}{1-u}\right)^{1/2} \left(a(u) + \frac{u+1}{\sqrt{2\pi}} b_1(u)\right) \equiv a^\gamma(u) \end{aligned} \quad (19)$$

The coordinate transformation  $x'(x, u)$  interchanges the cycles  $\gamma_i$  and reverses their orientation, therefore  $\gamma_{1'} = -\gamma_2$  and  $\gamma_{2'} = -\gamma_1$ . However, in spite of this and equation (17),  $a(\gamma(u))$  is not equal to  $a_D(u)$ . This is the mathematical manifestation of the failure of the full Montonen-Olive duality for  $N=2$  theories with a non-vanishing  $\beta$ -function. Moreover (19) is not even a symplectic change of special coordinates.

The previous description can be easily generalized to an arbitrary elliptic curve. In general, for each  $\gamma \in \Gamma_W$  we get

$$\frac{d\lambda^\gamma}{du} = \tilde{f}_\gamma(u) \lambda_1 \quad (20)$$

with

$$\tilde{f}_\gamma(u) = l(u) \frac{d\gamma(u)}{du} f_\gamma(u) \quad (21)$$

where  $f_\gamma(u)$  is determined by

$$\frac{d\lambda_1}{du} = f_\gamma(u) \lambda_1 \quad (22)$$

and  $l$  is the proportionality factor in (14) associated to the 1-form  $\lambda$  which gives the correct physical asymptotic behaviour. For instance,  $l = -\sqrt{2}/4\pi$  for the 1-form (16).

From the above expressions, we obtain

$$\begin{aligned} \begin{pmatrix} a_D \\ a \end{pmatrix} (\gamma(u)) &= \begin{pmatrix} \oint_{\gamma_{1'}} \lambda^\gamma \\ \oint_{\gamma_{2'}} \lambda^\gamma \end{pmatrix} (u) = \\ &= g_\gamma(u) \Gamma_\gamma \left[ \begin{pmatrix} a_D \\ a \end{pmatrix} (u) + h_\gamma(u) \begin{pmatrix} b_2 \\ b_1 \end{pmatrix} (u) \right] \equiv \Gamma_\gamma \begin{pmatrix} a_D^\gamma \\ a^\gamma \end{pmatrix} (u) \end{aligned} \quad (23)$$

with  $\Gamma_\gamma$  defined by (10) and where  $g_\gamma(u)$  and  $h_\gamma(u)$  are determined by (20) in terms of  $\tilde{f}_\gamma(u)$ . In particular it is easy to see that  $f_\gamma = g_\gamma^{-1}$ .

The special property of the elements  $\gamma \in \Gamma_W$  corresponding to global symmetries is that for them  $a^\gamma = a$  and  $a_D^\gamma = a_D$ , i.e. they lift to the bundle as good transformations in  $Sl(2; Z)$ <sup>6</sup>. Before going into a more detailed analysis of the transformations (23), let

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<sup>6</sup>As was already pointed out in [2], equation (2) defines the sections  $(a, a_D)$  up to a sign. This ambiguity appears explicitly when we express the elliptic modulus  $\tau(\gamma(u))$  in terms of  $\tau(u)$ . If we strictly use (2) and (18) we get that, while  $\tau(u) \in H^+$ ,  $\tau(\gamma(u)) \in H^-$ , where  $H^\pm$  are respectively the upper and lower half complex plane

$$\tau \rightarrow \frac{a'\tau + b'}{c'\tau + d'} \quad , \quad \Gamma'_\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad (24)$$

The matrix  $\Gamma'_\gamma$  does not belong to  $Sl(2; Z)$ , but satisfies  $(\Gamma'_\gamma)^2 = 1$  as a consequence of the fact that  $\gamma$

us make the following comment on the interplay between strong-weak coupling duality and scale invariance. We will consider as an example  $SU(2)$  SQCD with  $N_f = 1$ . In the massless case there exists three singularities  $(p_1, p_2, p_3)$  related by a global  $Z_3$  symmetry [3]. When a mass term for the quark is added, one of these singularities, let us say  $p_1$ , moves continuously with the mass to  $\infty$ , while the others become the singularities of the  $N_f = 0$  theory. For finite mass, the element  $\gamma \in \Gamma_W$  relating  $p_1$  with the singularities  $(p_2, p_3)$  transforms the holomorphic sections  $(a, a_D)$  in the way described by (23) (plus a constant shift due to the fact that, when a mass term is present, the 1-form  $\lambda$  has non-vanishing residues [3]). Geometrically this just means that the monodromy  $M_1$  around  $p_1$  will not be  $T$ -conjugated of the monodromies  $M_2$  or  $M_3$ . What this teaches us is that a finite mass breaks the global  $U(1)_{\mathcal{R}}$  symmetries, as they are represented in  $\Gamma_W$ , in formally the same way as the non-vanishing  $\beta$ -function, i.e. a non-zero scale  $\Lambda$ , breaks the Montonen-Olive duality, namely inducing on the holomorphic sections  $(a, a_D)$  changes of the type (23) with  $h_\gamma \neq 0$ . This fact strongly indicates that, at least for SUSY gauge theories, a necessary condition for duality will be to have, in addition to scale invariance, a non-anomalous  $U(1)_{\mathcal{R}}$ -symmetry<sup>7</sup>. In a different language, equation (23) reflects the dependence of the four dimensional physics on the geometry of the elliptic curve, i.e. the way it changes for two points  $u, \gamma(u)$  which describe the same moduli.

4. *Coupling to gravity.* A natural way to try to make sense of equation (23) is considering the coupling to gravity. Intuitively we can think of (23) as a  $Sp(4; Z)$  transformation by interpreting the extra piece in the periods  $b_i$ , as contributions from the gravitational sector associated to the additional  $U(1)$  field present in  $N=2$  supergravity: the graviphoton. Due to the presence of the graviphoton, it is necessary to introduce a new (non-dynamical) special coordinate  $(a_0, a_{D0})$  and to define the special manifold (quantum moduli) projectively. In this picture the transformation from  $u$  to  $\gamma(u)$ ,  $\gamma \in \Gamma_W$ , will become a good element in  $Sp(4; Z)$  if at the same time we perform a, in general singular,  $U(1)$  gauge transformation of the Kähler-Hodge line bundle which we have naturally defined when we pass from rigid to non-rigid special geometry<sup>8</sup>[16].

More precisely, denoting  $V = (a, a_D)$ , the rigid special geometry for a 1-dimensional permutes two singularities. In order to recover a positive  $\tau(\gamma(u))$ ,  $a$  or  $a_D$  should be redefined in a sign. Then equation (10) is verified,  $\Gamma_\gamma \in Sl(2; Z)$  being the matrix appearing in (23). Notice that this is already evident from (19), where we obtain  $a \rightarrow a_D, a_D \rightarrow a$  which differs in a sign from an  $S$  transformation.

<sup>7</sup>The existence of non-anomalous  $U(1)_{\mathcal{R}}$  symmetries together with scale invariance implies that even for  $N=1$  SUSY theories, the conformal phase shares many aspects of  $N=2$  theories. This fact is crucial in the  $N=1$  duality between  $SU(N_c)$  and  $SU(N_f - N_c)$  with  $N_f$  quarks [12].

<sup>8</sup>Another reason supporting this idea comes from Landau-Ginzburg theories. For Landau-Ginzburg models, it is possible to build all gravitational descendant fields inside the matter sector [13]. This allows us to interpret the reparametrizations of the superpotential  $W$  as contributions from gravitational descendants [14, 15]. Therefore transformations (23) should admit a natural representation when gravity is turned on, i.e. they should be elements of  $Sp(4; Z)$ .

moduli space, is defined by

$$\begin{aligned} d_u V &= U \\ D_u U &= C_{uuu} G_{u\bar{u}}^{-1} \bar{U} \\ d_u \bar{U} &= 0 \end{aligned} \quad (25)$$

where  $G_{u\bar{u}} = \text{Im}\tau(u)$  is the metric over the moduli space, the Yukawa coupling  $C_{uuu}$  is given by

$$C_{uuu} = \frac{d\tau}{du} \left( \frac{da}{du} \right)^2 \quad (26)$$

and the covariant derivative is

$$D_u = d_u - \Gamma_u \quad , \quad \Gamma_u = G_{u\bar{u}}^{-1} (d_u G_{u\bar{u}}) \quad (27)$$

For each  $\gamma \in \Gamma_W$  we define  $V^\gamma = (a^\gamma, a_D^\gamma)$  with  $a^\gamma, a_D^\gamma$  given by (23). It is easy to verify that  $V^\gamma$  satisfy (25) with

$$\begin{aligned} D_u^\gamma &= D_u - d_u \ln \tilde{f}_\gamma \\ U^\gamma &= \tilde{f}_\gamma U \end{aligned} \quad (28)$$

where  $\tilde{f}_\gamma$  is given by (21) and the quantities  $C_{uuu}^\gamma, G_{u\bar{u}}^\gamma$  are defined according to (26) and (27) in terms of  $V^\gamma$  instead of  $V$ . This modification of  $D_u$  is not allowed in rigid special geometry, which is showing again that changing from  $V$  to  $V^\gamma$  is not a symplectic transformation. However (28) can be naturally interpreted from the point of view of non rigid special geometry, whose defining relations are

$$\begin{aligned} D_u V &= U \\ D_u U &= e^K C_{uuu} G_{u\bar{u}}^{-1} \bar{U} \\ d_u \bar{U} &= G_{u\bar{u}} \bar{V} \\ d_u \bar{V} &= 0 \end{aligned} \quad (29)$$

with  $V = (a_0, a_1, a_{D1}, a_{D0})$  and the covariant derivative

$$D_u = d_u - G_{u\bar{u}}^{-1} (d_u G_{u\bar{u}}) + d_u K \quad (30)$$

where the piece  $d_u K$ ,  $K(u, \bar{u})$  being the Kähler potential, is the  $U(1)$  connection associated to the Hodge line bundle over the quantum moduli, present when we couple to gravity. Notice that the vector  $U$  now acquires a non-holomorphic part

$$U = d_u V + d_u K V \quad (31)$$

which is at the origin of the third equation in (29).

From (28) we observe that working with the sections  $(a^\gamma, a_D^\gamma)$  amounts, at the level of the vector  $U, \bar{U}$ , to multiplying by a global factor. In the framework of special geometry, this can be interpreted as a change in the projective coordinate

$$a_0 = 1 \rightarrow a_0^\gamma = \tilde{f}_\gamma \quad (32)$$



or, equivalently, as the gauge transformation

$$K \rightarrow K - \ln \tilde{f}_\gamma - \ln \bar{\tilde{f}}_\gamma \quad (33)$$

and therefore as a change in the covariant derivative (30) of the form required by (28). In special geometry, we can now define  $\lambda^\gamma$  by the equation

$$D_u^\gamma \lambda^\gamma = \tilde{f}_\gamma \lambda_1 \quad (34)$$

with  $D_u^\gamma$  defined by (28.1) and (30). Equation (34) is motivated by the first relation (29). We now define the sections  $a^\gamma$  and  $a_D^\gamma$  by the corresponding integrals around 1-cycles of the solution to (34). They satisfy  $a^\gamma = \tilde{f}_\gamma a$ ,  $a_D^\gamma = \tilde{f}_\gamma a_D$ , and therefore the action of  $\gamma$  on the special geometry coordinates is given by

$$\begin{pmatrix} a_{D0}^\gamma \\ a_{D1}^\gamma \\ a_1^\gamma \\ a_0^\gamma \end{pmatrix} = \tilde{f}_\gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{D0} \\ a_{D1} \\ a_1 \\ a_0 \end{pmatrix} \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Gamma_\gamma \quad (35)$$

with  $\Gamma_\gamma$  given by (10). Equation (35) implies that the action of  $\Gamma_W$  can be represented, once we couple to gravity, by an element of  $Sp(4; Z)$  plus the Kähler gauge transformation (33).

Notice that in rigid special geometry, the action of  $\gamma$  on  $(a, a_D)$  was defined by the condition  $a^\gamma(u) = a(\gamma(u))$ ,  $a_D^\gamma(u) = a_D(\gamma(u))$ , which was at the origin of the non-symplectic transformations (23). It is important to analyse to what extent this condition is verified by the non-rigid  $\gamma$  transformations (35). Let us consider an element  $\gamma \in \Gamma_W$  that interchanges the singularity at  $\infty$  with a finite singular point, say  $p_1$ , while leaving the rest fixed. In the case of zero masses for the quarks, the asymptotic behaviour of the sections  $(a, a_D)$  at  $\infty$  is given by [2, 3]

$$a(u') = \frac{1}{2} \sqrt{2u'} \quad , \quad a_D(u') = i \frac{k_\infty}{4\pi} \sqrt{2u'} \ln \frac{u'}{\Lambda^2} \quad (36)$$

At the singular point  $p_1$  some particle in the spectrum becomes massless. Using the dual description and up to an  $Sl(2; Z)$  rotation, the special coordinates behave as

$$a(u) = c_0(u - p_1) \quad , \quad a_D(u) = c_1 - \frac{ik_1}{2\pi} a \ln(u - p_1) \quad (37)$$

with  $c_0, c_1$  constants. Comparing the two limits, equation (23) implies<sup>9</sup>

$$g_\gamma(u) \sim \sqrt{u'} \quad (38)$$

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<sup>9</sup>In fact, from (23) we get  $g_\gamma h_\gamma \sim \sqrt{u'}$  instead of (38). However it can be seen that the function  $h_\gamma$  is regular at  $u = p_1$  and therefore it is enough to consider (38) in the following computations.

The map  $u' = \gamma(u)$  is, of course, singular at  $u = p_1$ . Taking into account only its singular part, and for a certain constant  $C$ , we have

$$u' = C(u - p_1)^{-k} \quad , \quad k > 0 \quad (39)$$

where  $k$  is determined again from (23), by correctly reproducing the monodromy at  $\infty$

$$k = k_1/k_\infty \quad (40)$$

Substituting now (38), (39) in the expression of the function  $\tilde{f}_\gamma$ , we obtain

$$\tilde{f}_\gamma = \frac{du'}{du} g_\gamma^{-1} \sim \frac{\sqrt{u'}}{u - p_1} \quad (41)$$

Therefore the special coordinates  $a_1^\gamma, a_{D1}^\gamma$  defined by (35) have the expected asymptotic behaviour at  $u \rightarrow p_1$ , namely they tend to  $a(\gamma(u)), a_D(\gamma(u))$ . Notice also that, if  $V = (a_0, a_1, a_{D1}, a_{D0})$  satisfy the special geometry relations (29), so does the vector  $V^\gamma$  defined by (35), with  $U^\gamma = \tilde{f}_\gamma U$  as we should expect from (28.1).

In our previous construction, the extra special coordinate  $(a_0, a_{D0})$  associated with the graviphoton plays a role similar to that of the mass in SQCD. In fact when a mass term is added, there appear monodromies which are not in  $Sp(2; Z)$  ( $v \rightarrow Mv + c, M \in Sl(2; Z)$ ). These monodromies have perfect sense once we formally treat the mass as a field [3]. In the case of non-vanishing  $\beta$ -function we are trying to give sense to the strong-weak duality transformations  $\gamma \in \Gamma_W$  as element in  $Sp(4; Z)$  by including as an extra degree of freedom the graviphoton multiplet.

*5. Duality and  $\sigma$ -model anomalies.* From physical grounds we should expect that if the  $N=2$  theories we are working with are some low-energy limit of a string theory, then the stringy effects will be able to restore the whole duality invariance. The picture that emerges from our previous construction seems to indicate a possible way to achieve this goal.

In  $N=2$  SUGRA, and this is specially clear when we formulate the theory starting with conformal supergravity and passing later to Poincaré supergravity, the projective coordinate  $a_0$  is not a real degree of freedom. Equivalently, the chiral  $U(1)$  gauge field  $A_\mu$  of the Weyl supermultiplet<sup>10</sup> is an auxiliary field that can be eliminated by solving the constraints in the same way as we are used to do in non-linear  $\sigma$ -models. Up to fermionic

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<sup>10</sup>The Weyl supermultiplet appears in the context of conformal gravity. It contains [17] the gauge fields associated with the superconformal symmetries, namely general coordinates and local Lorentz transformations, dilatations, special conformal boosts and local supersymmetries. In addition, for  $N=2$ , there exists a local chiral  $SU(2)$  and  $U(1)$ . Notice that this chiral  $U(1)$  is the gauge symmetry defining the Kähler-Hodge line bundle of special geometry.

terms, the field  $A_\mu$  can be expressed in terms of the Kähler potential as follows

$$A_\mu = \frac{i}{2}(\partial_\mu z (dz/du)^{-1} d_u K + \partial_\mu \bar{z} (d\bar{z}/d\bar{u})^{-1} d_{\bar{u}} K) \quad (42)$$

where  $z = a_1/a_0$  is the homogeneous special coordinate. The transformation (33) over the Kähler potential can be interpreted as a gauge transformation on  $A_\mu$ , which corresponds to passing from a coordinate patch in the quantum moduli space around  $u = p_1$ , to a local coordinate patch  $u' = \gamma(u)$  around  $\infty$ , for  $\gamma \in \Gamma_W$ . This transformation is characterized by the parameter  $k$  in (39), which is different from zero as a consequence that  $\Gamma_\gamma$  is not in the abelian subgroup generated by 1 and  $T$ .

As we have shown in the previous paragraph, once we use non rigid special coordinates we get for the action of  $\Gamma_\gamma$  the representation (35), which in particular means that if  $\tau(\gamma(u)) = \frac{-1}{\tau(u)}$  then  $a(\gamma(u)) = \tilde{f}_\gamma a_D(u)$ , i.e. strong-weak coupling duality if the special coordinates  $a$  and  $\tilde{f}_\gamma a$  can be considered as gauge equivalent. The fact that the gauge transformation (33) defined by  $\tilde{f}_\gamma$  is in general singular can be at the origin of a topological obstruction to mod by  $\Gamma_W$  of the type found in  $\sigma$ -model anomalies [18]. Using the field  $A_\mu$ , equation (33) and assuming that the quantum corrections corresponding to integrating over the fermions have already been taken into account in the geometry, singularities, of the quantum moduli, we can use, for a compactified quantum moduli, the following quantity as an indication of this topological obstruction<sup>11</sup>

$$\nu = \frac{1}{2\pi i} \oint_{C_\infty} d \ln \tilde{f}_\gamma = 1 + \sum_i \frac{k_i}{2k_\infty} \quad (43)$$

where  $C_\infty$  encircles the singularity at  $\infty$  and the coefficients  $k_\infty, k_i$  are given in (36) and (37) respectively. Equation (43) was obtained up to the normalization factors which can be derived from (42). The sum in (43) is over the singular points at which the function  $\tilde{f}_\gamma$  has a pole, namely  $\gamma(p_i) = \infty$ <sup>12</sup>.

The quantity (43) is showing the existence of an anomaly to define the theory on the moduli space of complex structures of the curve, i.e. to mod by the action of  $\Gamma_W$ . Once we have interpreted (43) as an anomaly, it will be natural to look for some compensating WZ term. This is in general not possible for  $\sigma$ -model anomalies [18]. In our case, the introduction of the dilaton will, very likely, play that role. In fact, the anomaly (43) is heuristically indicating that  $a_0$  can't be globally gauged away, which is strongly asking

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<sup>11</sup>For  $\sigma$ -model anomalies, the topological obstruction is defined by  $\int_{S^2 \times S^d} \hat{\phi}^* Ch(TM)$ , with  $d$  the dimension of the space-time and  $TM$  the tangent bundle to the  $\sigma$ -model manifold. The map  $\hat{\phi}$  should define a non-contractible two-parameter family of  $\sigma$ -model configurations. In our case, heuristically, it is the compactified  $u$ -plane and their singularities that define the "non-contractible" two-sphere, determining the reparametrizations and in this way the configuration of the auxiliary gauge field  $A_\mu$  on the quantum moduli.

<sup>12</sup>It is worth recalling that equations (38-41) and (43) could be different for an element  $\gamma \in \Gamma_W$  mapping finite singular points between themselves.

for an extra scalar degree of freedom in the gravity sector. We hope to address these problems in more detail elsewhere.

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